Section 7.8 Improper Integrals

1)

$$\int_0^\infty \frac{1}{1+x^2}dx = \lim_{a\to\infty}\int_0^a \frac{1}{1+x^2}dx = \lim_{a\to\infty}\left[\arctan x\right]_0^a = \lim_{a\to\infty}\arctan a - \arctan 0 =$$

$$\frac{\pi}{2} - 0 = \frac{\pi}{2}$$

2) Note I changed the lower limit to 1, otherwise the integral is divergent.

$$\int_1^\infty \frac{\ln x}{x^2}dx = \lim_{a\to\infty}\int_1^a \frac{\ln x}{x^2}dx = \lim_{a\to\infty}\left[\frac{-\ln x}{x} - \frac{1}{x}\right]_1^a = \lim_{a\to\infty} -\frac{\ln a}{a} - \frac{1}{a} - (0-1)$$

Clearly $\lim_{a\to\infty}\frac{1}{a} = 0$.

We use L'Hospital's rule to value $\lim_{a\to\infty}\frac{\ln a}{a}$ and find that it too is 0, so

$$\int_1^\infty \frac{\ln x}{x^2}dx = 1$$

3)

$$\int_e^\infty \frac{1}{x\ln x}dx = \lim_{a\to\infty}\int_e^a \frac{1}{x\ln x}dx = \lim_{a\to\infty}\left[\ln\left|\ln x\right|\right]_e^a = \lim_{a\to\infty}\left[\ln\left|\ln a\right| - \ln\left|\ln e\right|\right] =$$

$$\lim_{a\to\infty}\left[\ln\left|\ln a\right| - 0\right] = \lim_{a\to\infty}\ln\left|\ln a\right|$$

But $\lim_{a\to\infty}\ln\left|\ln a\right|$ increases without bound so the integral is Divergent

4)

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0}\int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0}\left[2\sqrt{x}\right]_a^1 = \lim_{a\to 0}\left(2\sqrt{1} - 2\sqrt{a}\right) = 2$$

5) Note I changed the upper limit to 1 so that the problem would make sense.

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}dx = \lim_{a\to 1}\int_0^a \frac{x}{\sqrt{1-x^2}}dx = \lim_{a\to 1}\left[-\sqrt{1-x^2}\,\right]_0^a = \lim_{a\to 1}\left(-\sqrt{1-a^2} - {}^{-}\sqrt{1}\right) = 1$$

6)

$$\int_0^{\infty} \frac{dx}{x(\ln x)^2}dx = \int_0^{.5} \frac{dx}{x(\ln x)^2}dx + \int_{.5}^{1} \frac{dx}{x(\ln x)^2}dx + \int_1^{\infty} \frac{dx}{x(\ln x)^2}dx$$

Examine just the middle term

$$\int_{.5}^{1} \frac{dx}{x(\ln x)^2}dx = \lim_{a\to 1}\int_{.5}^{a} \frac{dx}{x(\ln x)^2}dx = \lim_{a\to 1}\left[-\frac{1}{\ln x}\right]_{.5}^{a} =$$

$$\lim_{a\to 1}\left[-\frac{1}{\ln a} - \frac{-1}{\ln .5}\right]$$

But since $\frac{1}{\ln 1} = \frac{1}{0}$ the integral is Divergent.

7)

$$\int_0^1 x\ln x\,dx = \lim_{a\to 0}\int_a^1 x\ln x\,dx = \lim_{a\to 0}\left[\frac{x^2\ln x}{2} - \frac{x^2}{4}\right]_a^1 = \lim_{a\to 0}\left[0 - \frac{1}{4} - \left(\frac{a^2\ln a}{2} - \frac{a^2}{4}\right)\right] =$$

$$-\frac{1}{4} - \lim_{a\to 0}\frac{a^2\ln a}{2}$$

Using L'Hospital's rule we find the above limit goes to 0, so

$$\int_0^1 x\ln x\,dx = -\frac{1}{4}$$