Lesson Plan 13 - Partial Fractions 7.5

Integration using Partial Fractions

We've done some simpler versions of these before, eg:

Example 1:

$$\int \frac{x}{1+x}\,dx = \int 1 - \frac{1}{1+x}\,dx$$

What about something like

Example 2:

$$\int \frac{5x-4}{2x^2+x-1}\,dx \quad \text{Note that } \frac{d}{dx}2x^2+x-1 \neq A(5x-4)$$

Factoring the denominator

$$2x^2+x-1 = (2x-1)(x+1)$$

So we set $\int \frac{5x-4}{2x^2+x-1}\,dx = \int \frac{A}{2x-1} + \frac{B}{x+1}\,dx$

To find A and B we work the following:

$$\frac{A}{2x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(2x-1)}{(2x-1)(x+1)} = \frac{Ax+A+2Bx-B}{(2x-1)(x+1)} = \frac{x(A+2B)+(A-B)}{(2x-1)(x+1)}$$

Now we can see that $A+2B=5$ and $A-B=-4$

Subtracting the second from the first we have $3B=9$ or $B=3$ and therefore $A=-1$

$$\int \frac{5x-4}{2x^2+x-1}\,dx = \int \frac{-1}{2x-1} + \frac{3}{x+1}\,dx = -\frac{1}{2}\ln|2x-1| + 3\ln|x+1| = \ln\left|\frac{(x+1)^3}{\sqrt{2x-1}}\right| + C$$

Example 3:

$$\int \frac{3x^2+7x-2}{x^3-x^2-2x}\,dx$$

$$\frac{3x^2+7x-2}{x^3-x^2-2x}=\frac{3x^2+7x-2}{x(x+1)(x-2)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-2}$$

Next find the equations in *A,B,* and *C*

$$A(x+1)(x-2)+Bx(x-2)+Cx(x+1)=3x^2+7x-2$$

or

$$A\left(x^2-x-2\right)+B\left(x^2-2x\right)+C\left(x^2+x\right)=3x^2+7x-2$$

This shows us that

$$A+B+C=3$$
$$-A-2B+C=7$$
$$-2A=-2$$

This reduces our equations as follows:

$$A=1$$
$$B+C=2$$
$$-2B+C=8$$

These last two equations indicate that

$$B=-2$$
$$C=4$$

So

$$\int \frac{3x^2+7x-2}{x(x+1)(x-2)}\,dx=\int \frac{1}{x}+\frac{-2}{x+1}+\frac{4}{x-2}\,dx=\ln|x|-2\ln|x+1|+4\ln|x-2|+C= \quad =$$

$$\ln\left|\frac{x(x-2)^4}{(x+1)^2}\right|+C$$

Note that in this step:

$$A(x+1)(x-2)+Bx(x-2)+Cx(x+1)=3x^2+7x-2$$

if we set $x=0$ we get $A=1$
if we set $x=-1$ we get $B=-2$
and if we set $x=2$ we get $C=4$

Repeated Linear Factors

Note that if a term in the denominator is a repeated linear factor, you need an extra term.

Example 4:

$$\int \frac{x}{(x+2)^2(x-1)}\,dx = \int \frac{A}{(x+2)^2} + \frac{B}{x+2} + \frac{C}{x-1}\,dx$$

$$A(x-1) + B(x+2)(x-1) + C(x+2)^2 = x$$

Letting $x=1$ we find that $C = \frac{1}{9}$

Letting $x=-2$ we find that $A = \frac{2}{3}$

So $B = -\frac{1}{9}$

Irreducible factors:
Example 5:

Some factors are irreducible. In this case we proceed as follows:

$$\frac{1}{(1+x^2)(x-1)} = \frac{Ax+B}{1+x^2} + \frac{C}{x-1} \quad A = B = -\frac{1}{2}, \quad C = \frac{1}{2}$$

$$\int \frac{1}{(1+x^2)(x-1)}\,dx = \frac{1}{2}\int \frac{-(x+1)}{(1+x^2)} + \frac{1}{x-1}\,dx = \frac{1}{2}\int \frac{-x}{1+x^2} + \frac{-1}{1+x^2} + \frac{1}{x-1}\,dx =$$

$$\frac{-\ln(1+x^2)}{4} - \frac{\arctan x}{2} + \frac{\ln|x-1|}{2}$$