Lesson Plan 2 - Definite Integrals (5.2)

1) Take attendance, any new students?

We spoke on Tuesday about Riemann sums and how upper and lower sums are used to define the Riemann Integral

$$\int_a^b f(x)dx$$

This particular notation is called a **definite** integral. We will see the integral sign in three different contexts in the class.

1) A definite integral,eg: $\int_a^b f(x)dx$

Note that definite integral can be viewed as the area under a function on some interval. Also note that a definite integral is a number.

2) An indefinite integral is written as follows: $\int f(x)dx$

Note that I have removed the end point *a*, and *b*.

An indefinite integral is equal to the anti-derivative of the function.
Actually a family of functions related to the anti-derivative.
To see why, note the following:

$$\frac{d}{dx}x^3 = 3x^2$$

So the indefinite integral $\int 3x^2dx = x^3$

But note that $\frac{d}{dx}\left(x^3 + C\right) = 3x^2$ where C is any constant is also true.

So we write $\int 3x^2dx = x^3 + C$ where C can have any fixed value.

You wonder about how this affects what we were talking about on Tuesday where we found that:

$$\int_0^1 3x^2 dx = x^3\Big|_0^1 = 1 - 0 = 0$$

Well if we include the constant, we find that instead

$$\int_0^1 3x^2 dx = \left(x^3 + C\right)\Big|_0^1 = (1 + C) - (0 + C) = 1 + C - 0 - C = 1$$

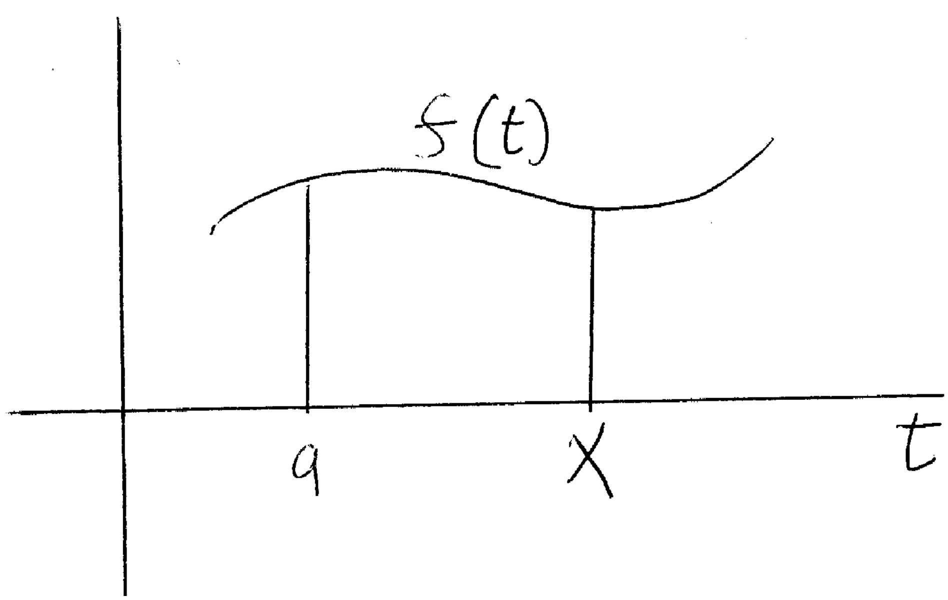
So the results are the same.

3) Finally, we might see an integral that looks like this:

$$A(x) = \int_a^x f(t)dt$$

You should note that the variable *t* here is a dummy variable that disappears when you evaluate the integral.

This turns out to be a function of *x*, which tells us the area under $f(x)$ from *a* to some *x*.



Note that this is neither a number, nor a family of functions, but instead is a single function.

Some examples we looked at last class suggest that

$$\int_a^b f(x)dx = \left[F(x)\right]_a^b \quad = F(b) - F(a)$$

$$Where\ F'(x) = f(x)$$

$$F(x)\, being\ the\ Anti-Derivative\ of\ \ f(x)$$

is a possible solution.
To show that this is the case, we proceed by defining a function as follows:

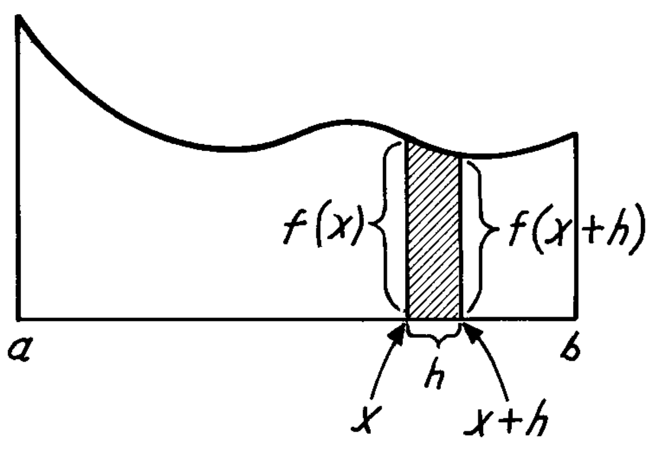
$$F(x) = \int_a^x f(t)\,dt$$

Recall that this is a function of x and not a definite integral.
It is a function which simply maps to the area under the curve $f(x)$ from the point a to the unknown point x.

Now consider this limit, which should look familiar:

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

What does this look like graphically?



$$F(x+h) = \text{area from } a \text{ to } x+h$$
$$F(x) = \text{area from } a \text{ to } x$$
$$F(x+h) - F(x) = \text{area from } x \text{ to } x+h$$
$$\frac{F(x+h) - F(x)}{h} = \frac{\text{area from } x \text{ to } x+h}{h} \overset{\text{approx}}{=} f(x) \text{ for small } h.$$

In this diagram you can see that as *h* --> ∞ the shaded area comes closer and closer to being a rectangle with area $\left[\frac{f(x+h)+f(x)}{2}\right]h$

As such

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{f(x+h)+f(x)}{2}=f(x)$$

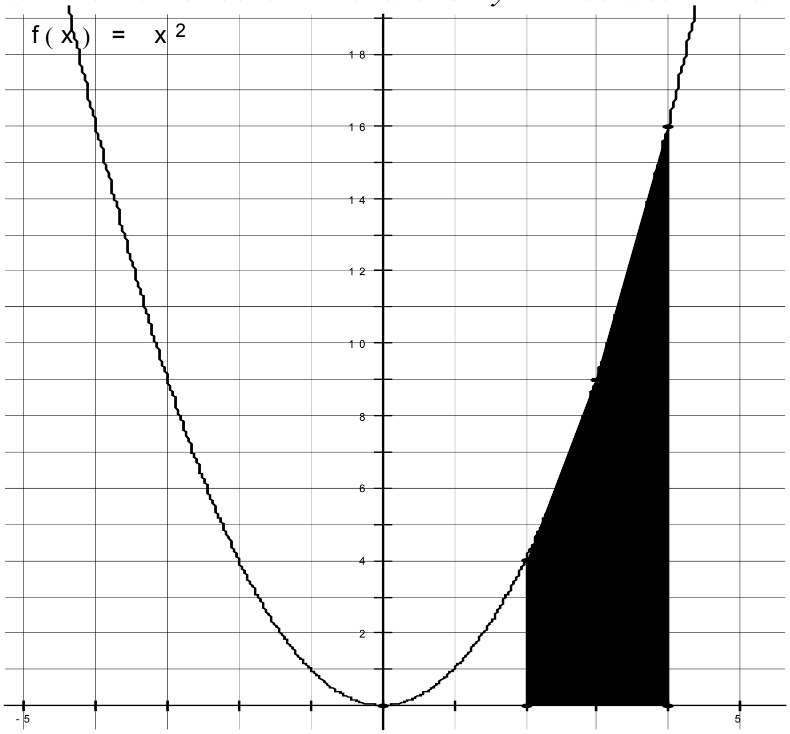
By the definition of the derivative, that means that

$$F'(x)=f(x)$$

That is $F(x)$ is the anti-derivative of $f(x)$

Let's let that settle in a bit with a few examples:

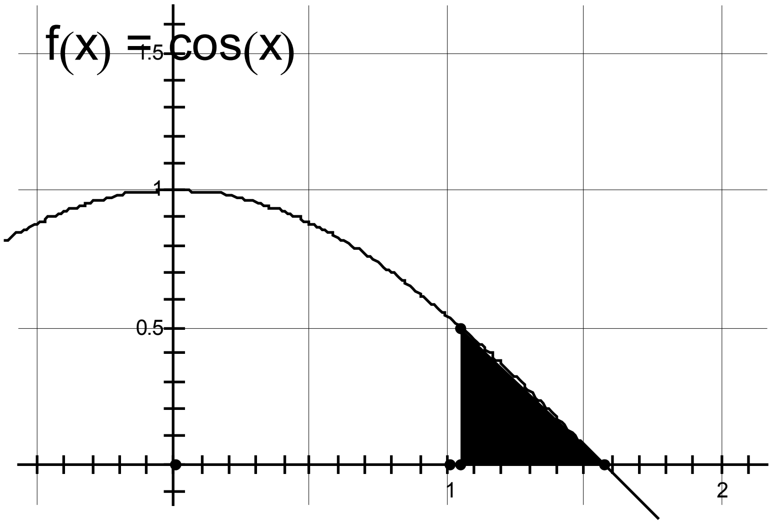
What is the area beneath the function $y = x^2$ between 2 and 4?



Since $\frac{d}{dx}x^3 = 3x^2$, the anti-derivative of x^2 is $F(x) = \frac{x^3}{3}$

Therefore $$\int_2^4 x^2 dx = \left[\frac{x^3}{3}\right]_2^4 = \frac{64}{3} - \frac{8}{3} = \frac{56}{3}$$

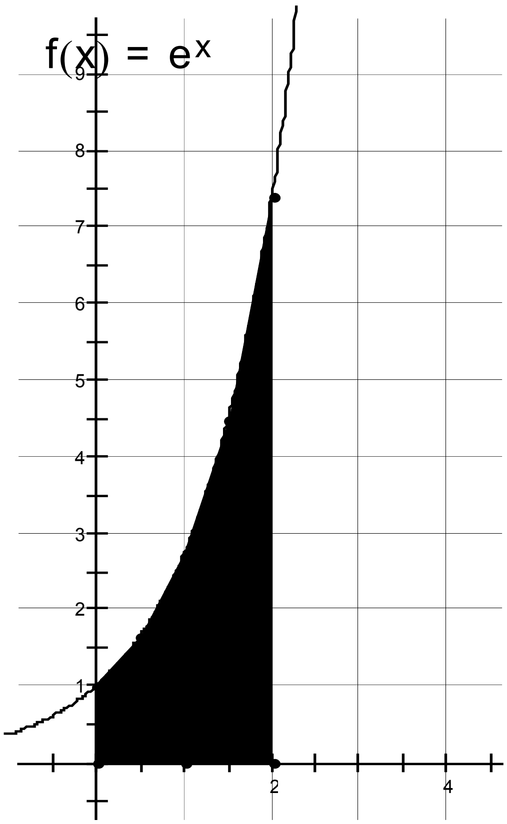
What is the area beneath the function $y = \cos(x)$ between $\frac{\pi}{3}$ and $\frac{\pi}{2}$?



Since $\frac{d}{dx}\sin(x) = \cos(x)$, the anti-derivative of $\cos(x)$ is $F(x) = \sin(x)$

$$\int_{\pi/3}^{\pi/2} \cos(x)\,dx = \left[\sin(x)\right]_{\pi/3}^{\pi/2} = \sin(\pi/2) - \sin(\pi/3) = \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} = \frac{\sqrt{2} - \sqrt{3}}{2}$$

What is the area beneath the function $y = e^x$ between 0 and 2?



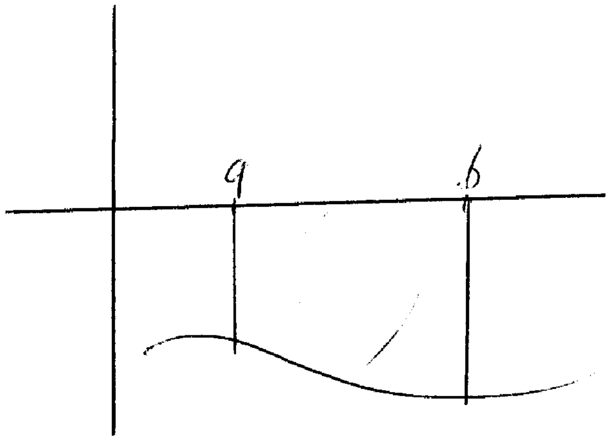
Since $\frac{d}{dx}e^x = e^x$, the anti-derivative of e^x is $F(x) = e^x$

$$\int_0^2 e^x dx = \left[e^x\right]_0^2 = e^2 - e^0 = e^2 - 1$$

[BREAK?]

[HANDOUT PROBLEM SHEET]

What happens now if our function is below zero?

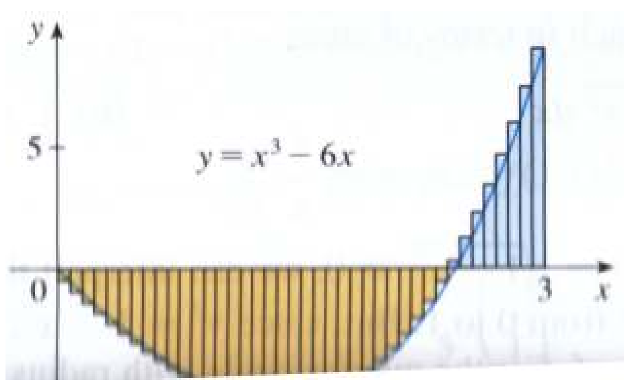


If we go back to our Riemann Sum definition

$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$$

We can see that the result of this sum is now negative.   It no longer represents the area, but it is the negative of the area between the function and y=0.

It is also possible that our function is both below and above the x axis.

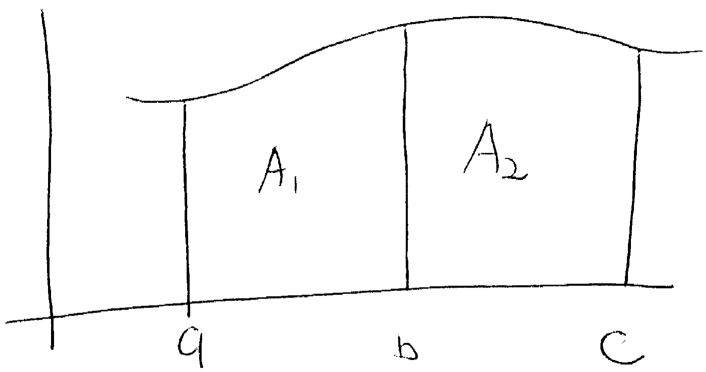


Here the definite integral might be positive or negative depending on the limits.

Let's take note of some properties of integrals.

1) Integration over continuous segments

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$



If we have $A_1 = \int_a^b f(x)\,dx$ and $A_2 = \int_b^C f(x)\,dx$

then is follows that since $A = A_1 + A_2$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

3) Integration in reverse

Now re-arranging the limits we can have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

But that means that:

$$A_1 = A_1 + A_2 + \int_c^b f(x)\,dx$$

or

$$A_2 = -\int_c^b f(x)\,dx$$

So if you reverse the order of integration, you reverse the sign of the integral.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

4) Integration on the interval $[a,a]$

We can now show that

$$\int_a^a f(x)\,dx = \int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^b f(x)\,dx - \int_a^b f(x)\,dx = 0$$

Which is probably what you would have expected.

5) integration of the sum and differences of functions

Looking back a Riemann sums we can see that

$$\sum_{i=1}^{n}\left[f\left(x_i^*\right)+g\left(x_i^*\right)\right]\Delta x_i=\sum_{i=1}^{n}f\left(x_i^*\right)\Delta x_i+g\left(x_i^*\right)\Delta x_i=\sum_{i=1}^{n}f\left(x_i^*\right)\Delta x_i+\sum_{i=1}^{n}g\left(x_i^*\right)\Delta x_i$$

Showing that

$$\int_a^b f(x)+g(x)\,dx=\int_a^b f(x)\,dx+\int_b^c g(x)\,dx$$

Substituting $-g(x)$ for $g(x)$ we get

$$\int_a^b f(x)-g(x)\,dx=\int_a^b f(x)\,dx-\int_b^c g(x)\,dx$$

6) Constants in integrals

This can be again shown using Riemann sums

$$\sum_{i=1}^{n}cf\left(x_i^*\right)\Delta x_i=c\sum_{i=1}^{n}f\left(x_i^*\right)\Delta x_i$$

so we have

$$\int_a^b cf(x)dx=c\int_a^b f(x)dx$$

6) Integrals of Absolute Values

$$\int_a^b |f(x)|\,dx$$

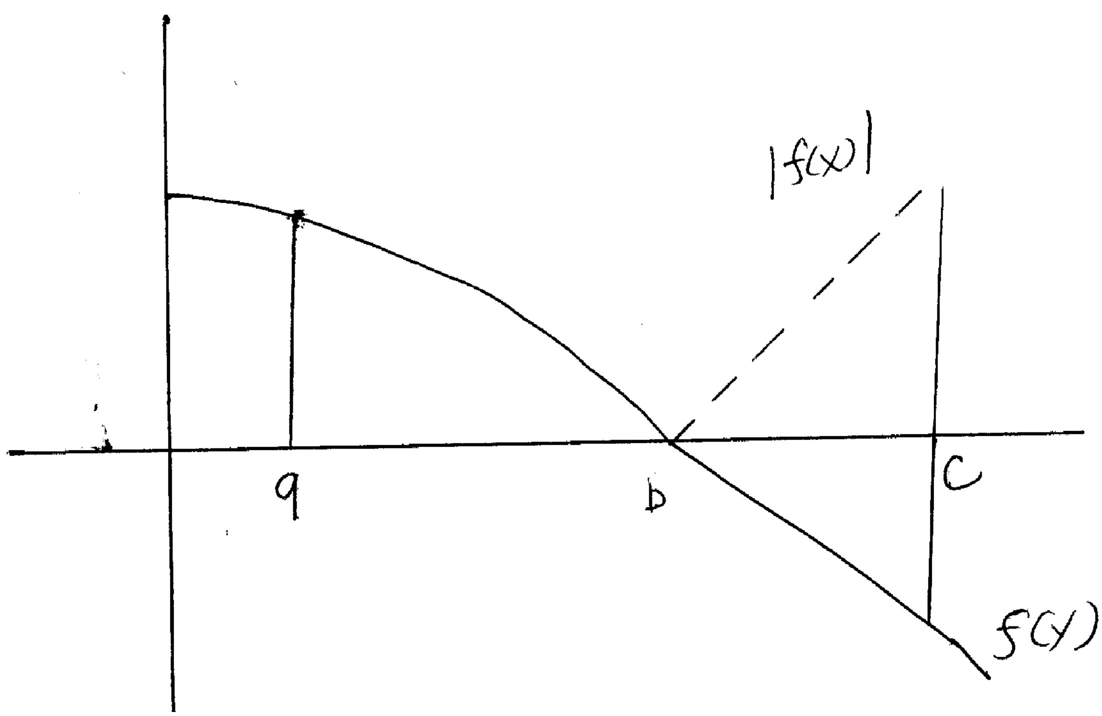
We can divide $[a,c]$ into sub-intervals where $f(x) \geq 0$ or $f(x) \leq 0$.

If $f(x) \geq 0$ on a sub-interval $[a',b']$ then $\int_{a'}^{b'} |f(x)|\,dx = \int_{a'}^{b'} f(x)\,dx$

If $f(x) \leq 0$ on a sub-interval $[a',b']$ then $\int_{a'}^{b'} |f(x)|\,dx = -\int_{a'}^{b'} f(x)\,dx$

So we can find the integral $\int_a^b |f(x)|\,dx$ by summing the sub-intervals.

Example:

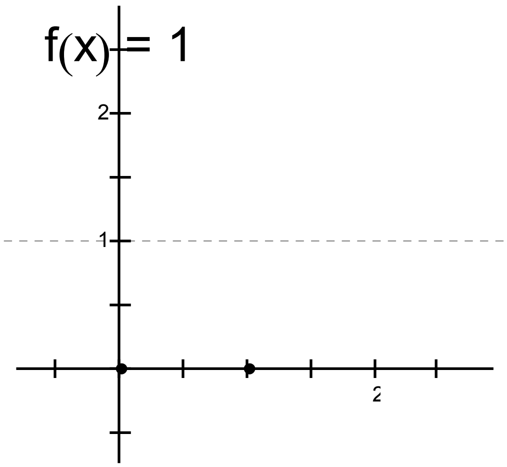


$$\int_a^c |f(x)|\,dx = \int_a^b |f(x)|\,dx + \int_b^c |f(x)|\,dx = \int_a^b f(x)\,dx - \int_b^c f(x)\,dx$$

Note that not all functions are integrable.

Example:

$$f(x)=\begin{cases}0 & x\in\mathbb{Q}\\ 1 & x\in\mathbb{Q}^c\end{cases}$$



A Condition for Riemannian Integrablity:

If $f(x)$ is a continuous on $[a,b]$ or is bounded on $[a,b]$ and has at most a finite number of discontinuties then $f(x)$ is integrable on $[a,b]$,

that is $\int_a^b f(x)\,dx$ exists!